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On the Particle Level of Galilean Quantum Field Theories

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Abstract

The procedure devised in a previous paper to obtain the exponential decay law for an unstable non-relativistic isolated particle within a Galilean Quantum Field Theory is applied to describe the transition from the particle to its decay products, as well as the transitions occurring in scattering processes.

1. *Introduction*

In two previous papers (Lanz, Lugiato & Ramella, 1973; Lanz *et al.*, 1974), to be referred to as A and B respectively, we treated extensively the problem of extracting from a Galilean Quantum Field Theory (GQFT) the exponential decay law for a non-relativistic unstable particle.† The characteristic feature of such an approach is the following: the distinction between the observed particle aspect and the unobserved fundamental field level is emphasised and the relation between the two levels is defined as an 'embedding' relation of the first one into the second. In such a passage from the field to the particle level, with the implied reduction of variables, irreversibility for the unstable particle arises, in strict analogy to what happens for a macroscopic system at the passage from the N-particle level to the macroscopic level.

In the case of stable particles the distinction between the field and the particle level need not be formalised into an explicit embedding relation, since in the Hilbert space $\mathscr{H}$ of the field model the vectors representing a single elementary particle are easily individuated in connection with an irreducible component of the representation of the Galilei (Poincaré) group and the vectors representing two or more stable particle states are individuated

† In recent years several contributions have been given to the problem of establishing the exponential decay law for unstable quantum systems. See Simonius (1970), Schulman (1970), Ekstein & Siegert (1971), Fonda *et al.* (to appear), Terent'ev (1972), Agodi *et al.* (1973), Rączka (preprint) and Lukierski (preprint).

by their asymptotic properties. Therefore the key point to individuate stable particles in QFT is the introduction of asymptotically stationary states. In a perturbative approach such states can be explicitly built in terms of bare particle states to yield suitable dressed states (Van Hove, 1955, 1956). In B we have used a rather natural generalisation of the eigenstates of the Hamiltonian $\bar{H}$ of the GQFT in the case of unstable particles. Such a procedure was intended to eliminate the basic ambiguity which one meets in the selection of the state vector for an unstable particle (Lévy, 1959). This ambiguity might be interpreted as a dependence of the time evolution of the unstable particle on its production; however, the very existence of a particle level requires that the particle must not exhibit any memory of its preparation. In a similar way, there is a macroscopic description (e.g. the hydrodynamical description of a fluid) in which there is no memory of the past; in the deduction of irreversible equations of motions (as the equations of hydrodynamics) the 'elimination' of the memory is the basic step introducing irreversibility. In conclusion, the description of an isolated particle can be rather clearly extracted from a GQFT, even when the particle is unstable. Of course, such a treatment can be quite naturally extended to a set of two or more particles, as is shown in Sections 2 and 3. However, one wants to extract from QFT the description of 'transitions' from a set of initial free particles to a (generally different) set of final free particles. This is necessary to give an account of the scattering processes, as well as of the decay products in a decay process. The usual S-matrix describes, in principle, all transitions between initial and final configurations involving only stable particles. As for transitions involving unstable particles, a partial answer is given by the 'damping theory' developed by Heitler and collaborators on the line indicated by Weisskopf and Wigner's treatment of the decay of an excited atom (Heitler, 1954; Weisskopf & Wigner, 1930; Goldberger & Watson 1964). Although such a formalism is able to give several features of the decay process, it is not satisfactory since it describes unstable particles by means of the bare eigenstates of a free Hamiltonian. Therefore the damping theory is only an approximate treatment of decaying states; e.g. the exponential decay law, which is indeed characteristic of the particle level (Schwinger, 1960), is obtained in the framework of the damping theory only in an approximate way.

The aim of the present paper is to apply the technique developed in B to the description of transitions. The results of B are slightly improved in Appendix 2.

It is shown in Section 3 that in the case of scattering between initial and final stable configurations such a technique provides the usual S-matrix of QFT. In Sections 4–6 we study the transition from an unstable particle to its decay products; the scattering processes involving unstable particles have been analysed in another paper (Casagrande & Lugiato, 1973). In Section 7 we give some explicit examples of the general analysis made in Sections 4–6; some concluding remarks are made in Section 8.

2. *Description of a Set of Particles*

To describe scattering and decay processes one has to consider systems of two or more particles. For the sake of simplicity, we shall treat only systems of two spinless particles; the extension to particles with spin and to many-particle systems is quite straightforward. Let us consider two particles O_1 and O_2 with mass M_1 and M_2 respectively; let us assume for the moment that O_1 and O_2 are not identical. According to A, such a two-particle system will be described in the space $\mathscr{H}_{12} \times R$ of couples (f, t) with $-\infty < t < +\infty, f \in \mathscr{H}_{12}$, $\|f\|_{\mathscr{H}_{12}} = 1$, where the Hilbert space $\mathscr{H}_{12}$ is the direct product of the Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ of O_1 and O_2 respectively, i.e. $\mathscr{H}_{12} = \mathscr{L}^2(R_3) \otimes \mathscr{L}^2(R_3)$. Using in R_6 the coordinates

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2, \qquad \mathbf{q} = \frac{M_1\mathbf{p}_2 - M_2\mathbf{p}_1}{M_1 + M_2} \tag{2.1}$$

we can write

$$\mathscr{H}_{12} = \int^{\oplus} h_{12}(\mathbf{P}, \mathbf{q}) \tag{2.2}$$

where $h_{12}(\mathbf{P}, \mathbf{q})$ is a one-dimensional Hilbert space subtended by the normalised element v. Therefore if $f \in \mathscr{H}_{12}, f = \{f(\mathbf{P}, \mathbf{q})v\}$, with $\int d\mathbf{P}\, d\mathbf{q}\, |f(\mathbf{P}, \mathbf{q})|^2 < \infty$. We shall use in the following the equivalent notation

$$\mathscr{H}_{12} = \int^{\oplus} h_{12}(\mathbf{P}) \tag{2.3}$$

$h_{12}(\mathbf{P}) = h_{12}^0$, h_{12}^0 being $\mathscr{L}^2(R_3)$; then if $f \in \mathscr{H}_{12}$, $f = f(\mathbf{P})$, where for every $\mathbf{P}$ $f(\mathbf{P}) \in \mathscr{L}^2(R_3)$.

We recall that in A and B we indicated by $\mathscr{G}_0$ the subgroup of the Galilei group of transformations $g = (0, \mathbf{a}, \mathbf{v}, R)$, $\mathbf{a} \in R_3$, $\mathbf{v} \in R_3$, $R \in SU(2)$; $\mathscr{H}_{12}$ is the space of a projective representation of $\mathscr{G}_0$ given by the direct product of the representations for O_1 and O_2, which have been defined in A.†

As in A, the physical content of the description is given by the expression

$$|(f_{\gamma'}, V_{12}(t - t_0)g_\gamma)_{\mathscr{H}_{12}}|^2, \qquad f_{\gamma'}, g_\gamma \in \mathscr{H}_{12}, \qquad \|f_{\gamma'}\| = \|g_\gamma\| = 1, \qquad t \geqslant t_0 \tag{2.4}$$

where γ' and γ are sharp properties respectively observed at time t and prepared at time t_0; the time evolution operator $V_{12}(t)$, $t \geqslant 0$, is linear and the multiplication rules of $V_{12}(-t)$, $t \leqslant 0$, with the generators of $\mathscr{G}_0$ are the same ones which hold for the time translation operator in the Galilei group.

If O_1 and O_2 are identical bosons (fermions), $\mathscr{H}_{12}$ is the Hilbert space of symmetrised (anti-symmetrised) elements of $\mathscr{H}_1 \otimes \mathscr{H}_2$. So far the treatment of the two particle system is quite the same as the treatment of the single particle. However, due to the interaction, the two particles can transform into other particles, so that one cannot consider the two-particle character of the system as an objective (i.e. observation independent) property of the system. Therefore $V(t)$ is in general not a semigroup for a system of two interacting particles.

† We stress that internal symmetries can be readily introduced in the present treatment defining the Hilbert space of particle O as the space of an irreducible representation of $\mathscr{G}_0 \otimes \tilde{\mathscr{G}}$ instead of $\mathscr{G}_0$, $\tilde{\mathscr{G}}$ being the group of internal symmetries.

We shall only require that, indicating by $V_1(t)$ and $V_2(t)$ the semigroups for the single particles,

$$|(f_{\gamma'}, V_{12}(t-t_0)g_\gamma)|^2 \simeq |(f_{\gamma'}, V_1(t-t_0)\otimes V_2(t-t_0)g_\gamma)|^2, \quad t_0 \leqslant t < t' \tag{2.5}$$

if the state $V_{12}(t-t_0)g_\gamma/\|V_{12}(t-t_0)g_\gamma\|_{\mathscr{H}_{12}}$ for $t_0 \leqslant t < t'$ is such that the two particles with large probability are so distant that the interaction is negligible.

3. *Embedding of a Set of Stable Particles*

One wants to embed the description given in the previous section into the GQFT (we refer the reader to B for the definition of the embedding relation). In this section we shall do this in the case of (two) stable particles, which is sufficient for the scope of the present paper; the extension to the case in which O_1 or O_2 or both are unstable is given by Casagrande & Lugiato (1973).

We embed the system $O_1 + O_2$ into the superselection sector $\bar{\mathscr{H}}_M$ with mass $M = M_1 + M_2$ of the GQFT which describes particles O_1 and O_2. This will be done by suitably defining the embedding operators $\overleftarrow{G}_-^{(1,2)}$ and $\overleftarrow{G}_+^{(1,2)}$ from $\mathscr{H}_{12}$ to $\bar{\mathscr{H}}_M$. To this extent we have recalled in Appendix 1 the definition of the 'asymptotically stationary states' (a.s.s.) given by Van Hove (1955, 1956) and Hugenholtz (1957) is the framework of a perturbative treatment. We shall be interested here in a.s.s. with one particle O_1 and one O_2, which we shall indicate as follows

$$\bar{u}^{(1,2)}_{\mathbf{p}_1\mathbf{p}_2,\text{a.s.}} = \bar{u}^{(1,2)}_{\mathbf{P},\mathbf{q},\text{a.s.}} = \bar{u}_{\mathbf{P}} \otimes \bar{u}_{\mathbf{q},\text{a.s.}} \tag{3.1}$$

where $\mathbf{P}$ and $\mathbf{q}$ have been defined in (2.1)

By translation invariance of GQFT we can write

$$\bar{\mathscr{H}}_M = \int^{\oplus} \bar{h}(\mathbf{P})\, d\mathbf{P}, \qquad \text{with} \qquad \bar{h}(\mathbf{P}) = \bar{h}^0$$

We shall consider in particular the subspace $\bar{h}^{0(1,2)}$ of $\bar{h}^0$ spanned by $\bar{u}^{(1,2)}_{\mathbf{q},\text{a.s.}}$. We define a linear embedding operator $\overleftarrow{G}^{(1,2)}$ from $\mathscr{H}_{12}$ to $\bar{\mathscr{H}}_M$ as follows. Let $f \in \mathscr{H}_{12}$, i.e. $f = \{f(\mathbf{P})\} = \{f(\mathbf{P}, \mathbf{q})v\}$, with $f(\mathbf{P}) \in h_{12}$, $\int d\mathbf{P}\, d\mathbf{q}\, |f(\mathbf{P}, \mathbf{q})|^2 < \infty$; then

$$\bar{f} = \overleftarrow{G}^{(1,2)} f = \{\overleftarrow{G}_0^{(1,2)} f(\mathbf{P})\}, \qquad \overleftarrow{G}_0^{(1,2)} f(\mathbf{P}) = \int d\mathbf{q} f(\mathbf{P}, \mathbf{q}) \bar{u}^{(1,2)}_{\mathbf{q},\text{a.s.}} \tag{3.2}$$

We assume the following 'strong' embedding relation (Lanz *et al.*, 1974) for the (1, 2)-particle system

$$(f, V_{12}(\tau)g)_{\mathscr{H}_{12}} = (\overleftarrow{G}^{(1,2)} f, e^{-i\bar{H}\tau} \overleftarrow{G}^{(1,2)} g)_{\bar{\mathscr{H}}_M} \qquad f, g \in \mathscr{H}_{12} \tag{3.3}$$

from which

$$(V_{12}(\tau)g)(\mathbf{P}, \mathbf{q}) = \int d\mathbf{P}'\, d\mathbf{q}' (\bar{u}^{(1,2)}_{\mathbf{P}\mathbf{q}\,\text{a.s.}}\, e^{-i\bar{H}\tau} \bar{u}^{(1,2)}_{\mathbf{P}'\mathbf{q}'\,\text{a.s.}})\, g(\mathbf{P}'\mathbf{q}')$$

From the transformation properties of the a.s.s. under the elements of $\mathscr{G}_0$ (see Appendix 1) one has immediately that symmetries are preserved by $\overleftarrow{G}^{(1,2)}$:

$$\overleftarrow{G}^{(1,2)}U(g) = \bar{U}(g)\overleftarrow{G}^{(1,2)}, \qquad g \in \mathscr{G}_0 \tag{3.4}$$

where $U(g)$ and $\bar{U}(g)$ are the unitary operators representing g in $\mathscr{H}_{12}$ and $\bar{\mathscr{H}}$ respectively. By (3.3) and (3.4), taking into account that $\exp(i\bar{H}\tau)$ represents time translation of τ in the GQFT, one has easily that $V_{12}(-\tau)$ has the expected multiplication rules with $U(g), g \in \mathscr{G}_0$. Further, if the model is symmetric under time inversion, also time inversion is preserved:

$$\overleftarrow{G}^{(1,2)}T = \bar{T}\overleftarrow{G}^{(1,2)} \tag{3.5}$$

T and $\bar{T}$ being the time-inversion operators in $\mathscr{H}_{1,2}$ and $\bar{\mathscr{H}}$ respectively. (3.5) follows from the property

$$\bar{T}\bar{u}^{(1,2)}_{\mathbf{P},\mathbf{q},\text{a.s.}} = \bar{u}^{(1,2)}_{-\mathbf{P},-\mathbf{q},\text{a.s.}} \tag{3.6}$$

(3.6) would not hold if to define $\overleftarrow{G}^{(1,2)}$ we had used incoming or outgoing states instead of a.s.s. Just by property (3.5) the embedding of a system of two *stable* particles is obtained by only one embedding operator $\overleftarrow{G}^{(1,2)}$: in the present case the two operators $\overleftarrow{G}_+$ and $\overleftarrow{G}_-$ introduced in B coincide. To discuss equation (2.5) in a mathematically precise way, let us consider states depending on a parameter l in such a way that in the limit $l \to \pm\infty$ the wave packets of the two particles in the x-description do not overlap: we define $g^{(l)}$ by

$$g^{(l)}(\mathbf{P}, \mathbf{q}) = g^{(0)}(\mathbf{P}, \mathbf{q}) \exp\left[-il\left(\frac{P^2}{2M} + \frac{q^2}{2\mu}\right)\right] = (V^{(1)}(l) \otimes V^{(2)}(l)g^0)(\mathbf{P}, \mathbf{q}) \tag{3.7}$$

Then one easily proves by (3.3), (A.5) and (A.7) that

$$\lim_{l \to \pm\infty} [(f^{(l)}, V_{12}(\tau)g^{(l)})_{\mathscr{H}_{12}} - (f^{(l)}, V^{(1)}(\tau) \otimes V^{(2)}(\tau)g^{(l)})_{\mathscr{H}_{12}}] = 0 \tag{3.7$'$}$$

A simple and well-expected asymptotic property of $V_{12}(\tau)$ holds by (A.6) and (A.7):

$$\lim_{\substack{t_0 \to -\infty \\ t \to +\infty}} (f^{(t)}, V_{12}(t - t_0)g^{(t_0)}) = \int d\mathbf{P}\, d\mathbf{q}\, d\mathbf{P}'\, d\mathbf{q}'\, f^{(0)}(^0\mathbf{P}, \mathbf{q})g^{(0)}(\mathbf{P}, \mathbf{q})$$

$$\cdot\, (\bar{u}^{(1,2)}_{\mathbf{P},\mathbf{q}\,\text{out}}, u^{(1,2)}_{\mathbf{P}',\mathbf{q}'\,\text{in}})_{\bar{\mathscr{H}}_M} \tag{3.8}$$

where $(\bar{u}^{(1,2)}_{\mathbf{P},\mathbf{q}\,\text{out}}, u^{(1,2)}_{\mathbf{P}',\mathbf{q}'\,\text{in}}) = S^{(1,2)}_{\mathbf{P},\mathbf{q};\mathbf{P}',\mathbf{q}'}$ is the S-matrix for the elastic scattering of particles (1.2) in GQFT.

Let us consider two different two-particles systems (1, 2) and (3, 4) and assume that each one can be embedded into $\bar{\mathscr{H}}_M$. Then we are immediately led to write down a probability amplitude for the transition from a state

$g^{(1,2)} \in \mathscr{H}_{12}$ prepared at time to a state $f^{(3,4)} \in \mathscr{H}_{34}$ observed at time t:

$$A(g^{(1,2)}, t_0 \to f^{(3,4)}, t) = (e^{i\bar{H}t}\overleftarrow{G}^{(3,4)}f^{(3,4)}, e^{i\bar{H}t_0}\overleftarrow{G}^{(1,2)}g^{(1,2)})_{\overline{\mathscr{H}}_M},$$
$$\|f^{(3,4)}\|_{\mathscr{H}_{3,4}} = \|g^{(1,2)}\|_{\mathscr{H}_{1,2}} = 1 \tag{3.9}$$

Introducing the states $g^{(1,2)(l)}, f^{(3,4)(l)}$ by definition (3.7) one easily has the expected relations:

$$\lim_{l \to \pm\infty} A(g^{(1,2)(l)}, t_0 \to f^{(3,4)(l)}, t) = 0 \tag{3.10}$$

$$\lim_{\substack{t_0 \to -\infty \\ t \to +\infty}} A(g^{(1,2)(t_0)}, t_0 \to f^{(3,4)(t)}, t) = \int d\mathbf{P}\, d\mathbf{q}\, d\mathbf{P}'\, d\mathbf{q}'.$$
$$f^{(3,4)(0)*}(\mathbf{P}, \mathbf{q})g^{(1,2)(0)}(\mathbf{P}', \mathbf{q}')(\bar{u}^{(3,4)}_{\mathbf{P},\mathbf{q},\,\text{out}}, \bar{u}^{(1,2)}_{\mathbf{P}',\mathbf{q}',\,\text{in}})$$
$$= (f^{(3,4)}S^{(3,4;1,2)}g^{(1,2)})_{\mathscr{H}_{3,4}} \tag{3.10'}$$

where $S^{(3,4;1,2)}$ is a linear operator from $\mathscr{H}_{12}$ to $\mathscr{H}_{3,4}$, defined as

$$(S^{(3,4;1,2)}g^{(1,2)}(\mathbf{P}, \mathbf{q}) = \int d\mathbf{P}'\, d\mathbf{q}'\, g^{(1,2)}(\mathbf{P}', \mathbf{q}')S^{(3,4;1,2)}_{\mathbf{P},\mathbf{q};\mathbf{P}',\mathbf{q}'} \tag{3.11}$$

$S^{(3,4;1,2)}_{\mathbf{P},\mathbf{q};\mathbf{P}',\mathbf{q}'}$ being the usual S-matrix element in GQFT. We see therefore that the embedding technique reproduces the usual unitary S-matrix. The embedding procedure in the case of stable particles substantially coincides with the elegant formulation of scattering in quantum field theory given by Guerra (1970). In his paper, the space $\mathscr{H}_{12} \otimes \mathscr{H}_{34} \otimes, \ldots$ is called asymptotic physical particle space, to be contrasted to the Hilbert space $\overline{\mathscr{H}}$ of the model called the bare particle space; the embedding operators $\overleftarrow{G}$ are called 'dressing operators' and indicated by T.

Defining generalised wave operators $W^{(\pm)}_{1,2}$ from $\mathscr{H}_{12}$ to $\overline{\mathscr{H}}$ as follows

$$W^{(\pm)}_{1,2} = s\text{-}\lim_{t \to \pm\infty} e^{i\bar{H}t}\overleftarrow{G}^{(1,2)}(V^{(1)}(t) \otimes V^{(2)}(t)) \tag{3.12}$$

where we assume that the strong limit exists, one has by (3.9), (3.7) and (3.10′) that

$$S^{(3,4;1,2)} = W^{(-)+}_{3,4}\, W^{(+)}_{1,2} \tag{3.13}$$

4. *Decay of an Unstable Particle*

In this and the following sections we shall use several symbols which have been introduced in B. For the sake of brevity, we shall assume familiarity with such notations. Let $\mathscr{H}$ be the Hilbert space of an unstable particle O with mass M, and $V(t)$ the one-parameter semigroup giving the time evolution of the isolated particle (exponential decay low). The main result of B has been to show that one can find two sequences of linear operators from $\mathscr{H}$ to $\mathscr{H}_M$ $\{\overleftarrow{G}_{(-)n}\}$, $\{\overleftarrow{G}_{(+)n}\}$, such that for any $f, g \in \mathscr{H}$†

$$(f, V(t-t_0)g)_{\mathscr{H}} = \lim_{\substack{n\to\infty \\ n'\to\infty}} (e^{i\bar{H}t} f_{n'}^{(-)}, e^{i\bar{H}t_0} \bar{g}_n^{(+)})_{\bar{\mathscr{H}}} \tag{4.1}$$

with

$$f_n^{(-)} = \overleftarrow{G}_{(-)n} f, \qquad g_n^{(+)} = \overleftarrow{G}_{(+)n} g, \qquad t \geqslant t_0 \tag{4.1'}$$

The sequences $\bar{f}_n^{(-)}, \bar{g}_n^{(+)}$ are in general not convergent.

Let $(1, 2)$ be the decay products of the unstable particle O (obviously $M = M_1 + M_2$). The element $g \in \mathscr{H}$, $\|g\|_{\mathscr{H}} = 1$ is a mathematical representation of a class of preparations of the unstable particle, i.e. to g all preparations correspond which lead to the same statistics for all 'single particle' observations represented by the elements $f \in \mathscr{H}$, $\|f\|_{\mathscr{H}} = 1$. There can be effects raised by the decay products which likewise show the same statistics for all preparations represented by g. Such effects are therefore independent of the production of O and of the preparation procedure of the initial state g. We want to identify the elements of $\mathscr{H}_{12}$ corresponding to such effects and calculate the probabilities of these effects. A very natural criterion to select at time t preparation independent effects is the following one: the sequence

$$A_n = (e^{i\bar{H}t} \overleftarrow{G}^{(1,2)} f^{(1,2)}, e^{i\bar{H}t_0} \bar{g}_n^{(+)})_{\bar{\mathscr{H}}} \tag{4.2}$$

must converge and $|A_n| \to A$ with $0 \leqslant A \leqslant 1$, for all $g \in \mathscr{H}$; then the transition probability is given by modulus squared limit:

$$P(g, t_0 \to f^{(1,2)}, t) = |\lim_{n\to\infty} A_n|^2 = A^2 \tag{4.3}$$

Let us call $\mathscr{E}$ the set of states $f^{(1,2)}$ corresponding to these two-particle preparation-independent effects.‡ Of course, with the definition (4.3) one cannot expect to find in general a unitarity relation for the sum of the decay probability and the probability to find the undecayed particle. In fact to establish such a unitarity one must know which effects of the decay products

† In B (4.1) was expressed in the equivalent form (A.2.12) of the present paper. Actually a stronger result than (4.1) holds, in fact the sequence in (4.1) converges uniformly with respect to $f, g \in \mathscr{H}$ and t for $t_0 \leqslant t < t_0 + T$, with T arbitrarily large. The proof is given in Appendix 2 of the present paper.

‡ One could consider more generally effects represented by bounded operators $0 \leqslant F \leqslant 1$ on $\mathscr{H}_{12}$ such that

$$\lim_{n\to\infty} \mathrm{Tr}((\overleftarrow{G}^{(1,2)} F \overleftarrow{G}^{(1,2)+}) P\bar{g}_n^{(+)}) = L_{F,g}, \text{ with } P\bar{g}_n^{(+)} = \bar{g}_n^{(+)}(\bar{g}_n^{(+)}, \cdot)_{\bar{\mathscr{H}}},$$

exists for all

$$g \in \mathscr{H}, \text{ and } 0 \leqslant L_{F,g} \leqslant 1$$

We consider here only those effects F which are projectors on $\mathscr{H}_{12}$. The general concept of 'effect' has been recently introduced by Ludwig (1970).

are compatible with a property of the undecayed particle; such a problem is outside the scope of the present paper. One might be inclined to expect that all properties of the undecayed particles are compatible with all effects of the decay products, then one would ask that the vectors of $\bar{\mathscr{H}}$ representing the unstable particle are orthogonal to all the vectors representing the decay products. In fact, this schematisation has been used by several authors (see e.g. Fonda & Ghirardi, 1972; Jersak, 1970). However, such a schematisation is oversimplified, since one cannot expect that it holds if the decay products are at a microscopic distance.

By definition (4.3) of transition probability the following problem arises: sequences $\{\overleftarrow{G}_{(+)n}\}$, $\{\overleftarrow{G}_{(-)n}\}$ are not uniquely determined by condition (4.1). This fact becomes relevant when (4.3) is considered, since we shall show that not all choices of such sequences lead to the same transition probability. Therefore we shall be obliged to introduce some reasonable criterion to select the 'right' class of sequences. To see this point carefully, let us recall how $\overleftarrow{G}_{(\pm)n}$ are built from generalised dressed states for unstable particles (Lanz *et al.*, 1974). Since O is spinless, we have (Lanz, Lugiato & Ramella, 1973) $\mathscr{H} = \int^{\oplus} h_0(\mathbf{P})\, d\mathbf{P}$ with $h_0(\mathbf{P}) = h_0$, h_0 being a one-dimensional space; let u, $\|u\|_{h_0} = 1$ span h_0. If $f = \{f(\mathbf{P})u\} \in \mathscr{H}$, $g = \{g(\mathbf{P})u\} \in \mathscr{H}$, one has

$$(f, V(t)g)_{\mathscr{H}} = \int d\mathbf{P} f^*(\mathbf{P}) g(\mathbf{P}) \exp\left[-i\left(\frac{P^2}{2M} + \lambda\right)t\right] \tag{4.4}$$

where $\lambda = U - i(\gamma/2)$, U being the internal energy and γ the inverse lifetime of O. One considers the z-dependent operators from $\mathscr{H}$ to $\bar{\mathscr{H}}_M$, holomorphic in z for $\operatorname{Im} z > 0$

$$\overleftarrow{G}_{(+)}(z)\,\{f(\mathbf{P})u\} = \{\overleftarrow{G}_{(+)0}(z) f(\mathbf{P})u\} \tag{4.5}$$

$$\overleftarrow{G}_{(+)0}(z)u = Z^{1/2}\,[\bar{P}_0 - \bar{N}_0(z)]\,\bar{u} \tag{4.5$'$}$$

$$\overleftarrow{G}_{(-)}(z) = \bar{\mathrm{T}}\overleftarrow{G}_{(+)}(z)T, \qquad \overleftarrow{G}_{(-)}(z)\{f(\mathbf{P})\bar{u}\} = \{\overleftarrow{G}_{(-)0}(z) f(\mathbf{P})\bar{u}\} \tag{4.6}$$

where $\bar{u}$ is a suitable element of $\bar{\mathscr{H}}_M$, $\bar{P}_0$ is an orthogonal projection operator on $\bar{h}^{(0)}$;† the linear operator $\bar{N}_0(z)$ on $\bar{h}_0$ is given by

$$\bar{N}_0(z) = (1 - \bar{P}_0)\frac{1}{(1 - \bar{P}_0)\bar{E}_0(1 - \bar{P}_0) - z}(1 - \bar{P}_0)\bar{E}_0\bar{P}_0 \tag{4.7}$$

where $\bar{E}_0$ is the internal energy operator in $\bar{h}_0$; we refer to B for the definition of the number Z and for further specifications on $\bar{u}$ and $\bar{P}_0$. One proves (Lanz *et al.*, 1974) that

† The projector technique has been used in the treatment of unstable particles and in related topics in Horwitz & Marchand (1971), Araki *et al.* (1957), Grecos & Prigogine (1972) and Agodi *et al.* (1973).

$$(f, V(t)g)_{\mathscr{H}} = \int d\mathbf{P} f^*(\mathbf{P}) g(\mathbf{P}) \exp\left(-i\frac{P^2}{2M}t\right) \frac{1}{(2\pi i)^2} \oint_C dz_1 \frac{1}{z_1 - \lambda}$$

$$\cdot \oint_{0^c} dz_2 \frac{1}{z_2 - \lambda} (\overleftarrow{G}_{(-)0}(z_1)u, \exp(-i\bar{E}_0 t)\overleftarrow{G}_{(+)0}(z_2)u)_{\bar{h}_0, \text{a.c.}} \tag{4.8}$$

where the suffix a.c. means that the scalar product, which is an analytic function of z_1, z_2 for $\operatorname{Im} z_1 > 0$, $\operatorname{Im} z_2 > 0$ must be analytically continued in z_1 and z_2 to reach the point $z_1 = z_2 = \lambda$; C is any circuit, taken counter clockwise and contained in the analyticity region of the integrand, around the point $z = \lambda$. If O is stable, one has further that

$$\overleftarrow{G}_{(+)}(U) = \overleftarrow{G}_{(-)}(U) \text{ def } \overleftarrow{G}$$

and

$$\bar{H}\{\overleftarrow{G}_0 f(\mathbf{P})u\} = \left\{\left(\frac{P^2}{2M} + U\right)\overleftarrow{G}_0 f(\mathbf{P})u\right\} \tag{4.9}$$

One sees therefore from (4.8) that the elements $\overleftarrow{G}_{(+)}(z)g$, $\overleftarrow{G}_{(-)}(z)f$ play the role of generalised dressed states for the unstable particle.

The sequences $\{\overleftarrow{G}_{(-)n}\}$, $\{\overleftarrow{G}_{(+)n}\}$ are built as linear combinations of derivatives of $\overleftarrow{G}_{(-)}(z)$, $\overleftarrow{G}_{(+)}(z)$, as is shown explicitly in Appendix 2.

As the right-hand side of (4.1) coincides with the right-hand side of (4.8), we expect that $\lim_{n\to\infty} A_n$ (see (4.2)) is given by the following expression:

$$A = \int d\mathbf{P} \exp\left(-i\frac{P^2}{2M}t\right) g(\mathbf{P}) \frac{1}{2\pi i} \oint_C dz \frac{1}{z - \lambda} (\overleftarrow{G}^{(1,2)} f^{(1,2)}(\mathbf{P}),$$

$$\exp(-i\bar{E}_0 t)\overleftarrow{G}_{(+)0}(z)u)_{\bar{h}_0, \text{a.c.}} \tag{4.10}$$

provided the scalar product in (4.10) is analytically continuable in z to reach the point $z = \lambda$. In fact, (4.10) will be proved in the following sections; the relevant point is that not all choices of the path C lead to the same value for the transition probability $|A|^2$.

To end this section, let us mention how one can build the probability that a particle O is produced in a collision between O_1 and O_2:

$$P(f^{(1,2)}, t_0 \to g, t) = |\lim_{n\to\infty} B_n|^2 \tag{4.11}$$

with

$$B_n = (e^{i\bar{H}t}\bar{g}_n^{(-)}, e^{i\bar{H}t_0}\overleftarrow{G}^{(1,2)} f^{(1,2)})_{\bar{\mathscr{H}}} \tag{4.12}$$

where

$$g \in \mathscr{H}$$

and

$$f^{(1,2)} \in \mathscr{E}.$$

5. *Some Technical Consideration*

(a) According to (4.10) let us consider the expression

$$(\bar{u}^{(1,2)}_{\mathbf{q},\text{a.s.}}, e^{-i\bar{E}_0 t}\overleftarrow{G}_{(+)0}(z)u)_{\bar{h}_0} \tag{5.1}$$

We call it $D(z, \omega, t)$ because by rotation invariance it depends on $\mathbf{q}$ only through $\omega = q^2/2\mu$. We recall the well-known operator identity (Lanz *et al.*, 1974)

$$e^{-i\bar{E}_0 t} = \frac{-1}{2\pi i} \int_{-\infty+i\epsilon}^{+\infty+i\epsilon} dz' e^{-iz't} \Bigg\{ [\bar{P}_0 - \bar{N}_0(z')] \frac{1}{z' + \bar{M}_0(z')} [\bar{P}_0 - \bar{N}_0^+(z'^*)]$$

$$+ (1 - \bar{P}_0) \frac{1}{z' - (1 - \bar{P}_0)\bar{E}_0(1 - \bar{P}_0)} (1 - \bar{P}_0) \Bigg\}, \qquad t > 0 \tag{5.2}$$

where $\bar{M}_0(z)$ is a linear operator in $\bar{h}_0$ given by

$$\bar{M}_0(z) = \bar{P}_0\bar{E}_0(1 - \bar{P}_0) \frac{1}{(1 - \bar{P}_0)\bar{E}_0(1 - \bar{P}_0) - z} (1 - \bar{P}_0)\bar{E}_0\bar{P}_0 - \bar{P}_0\bar{E}_0\bar{P}_0 \tag{5.3}$$

Then by (4.5′), the first resolvent identity and the further identity

$$\bar{N}_0^+(z^*)\bar{N}_0(z') = \frac{\bar{M}_0(z) - \bar{M}_0(z')}{z - z'} \tag{5.4}$$

one has quite easily that

$$D(z, \omega, t) = \frac{-1}{2\pi i} \int_{-\infty+i\epsilon}^{+\infty+i\epsilon} dz' \frac{e^{-iz't}}{z' - z} \Bigg(\bar{u}^{(1,2)}_{\mathbf{q},\text{a.s.}}, \Bigg\{ [\bar{P}_0 - \bar{N}_0(z)]$$

$$- [\bar{P}_0 - \bar{N}_0(z')] \frac{1}{z' + M_0(z')} [z + \bar{M}_0(z)] \Bigg\} \bar{u} \Bigg)_{\bar{h}_0}, \qquad t > 0 \tag{5.5}$$

$D(\omega, z, t)$ is analytic in z for all physical values of ω, when $\operatorname{Im} z > 0$. The first term in the brackets in (5.5) is the only critical one for the analyticity properties of $D(\omega, z, t)$ as a function of z when z approaches the real axes from above; in fact, the other term depends on z through $\bar{M}_0(z)$, the analyticity properties of which have been already considered in B. Let us consider the perturbative expansion of $\bar{u}^{(1,2)}_{\mathbf{q},\text{a.s.}}$ and $\bar{N}_0(z)$ in terms of the interaction; i.e. we put $\bar{E}_0 = \bar{E}_{0F} + \lambda\bar{E}_{0I}$, where $\bar{E}_{0I}$ is the interaction internal energy, and expand $u^{(1,2)}_{\mathbf{q},\text{a.s.}}$ and $\bar{N}_0(z)$ in powers of λ. We recall that the zeroeth term in the expansion of $\bar{u}^{(1,2)}_{\mathbf{q},\text{a.s.}}$ is the bare state $\bar{u}^{(1,2)}_{\mathbf{q},F}$, eigenstate of $\bar{E}_{0F}$ with eigenvalue

$U_{1,0} + U_{1,0} + \omega$, $U_{2,0}$ and $U_{3,0}$ being the internal energies of the bare particles O_1 and O_2 respectively. Then, since in all cases of interest

$$\frac{1}{(1-\bar{P}_0)\bar{E}_{0F}(1-\bar{P}_0)-z}(1-\bar{P}_0) = \frac{1}{\bar{E}_{0F}-z}(1-\bar{P}_0)$$

$$(1-\bar{P}_0)\bar{u}^{(1,2)}_{\mathbf{q},F} = \bar{u}^{(1,2)}_{\mathbf{q},F} \tag{5.6}$$

we see that the term $(\bar{u}^{(1,2)}_{\mathbf{q},F}, \bar{N}_0(z)\bar{u})$ contains the pole singularity $[U_{1,0} + U_{2,0}$ $\omega - z]^{-1}$. It is quite reasonable to expect that, taking account of the other terms in the expansion of $\bar{u}^{(1,2)}_{\mathbf{q},\text{a.s.}}$, such a pole is merely shifted to its 'dressed' value, so that we find the singularity $[U_1 + U_2 + \omega - z]^{-1}$, U_1 and U_2 being the internal energies of the dressed particles O_1 and O_2 respectively. We shall verify this fact on concrete examples in Section 7.

Let U be the internal energy of the unstable particle O, in general $U > U_1 + U_2$. Let $\omega_0 = U - U_1 - U_2$; we assume that there is a neighbourhood I_δ of ω_0, $\omega_0 - \delta < \omega < \omega_0 + \delta$, such that for every $\omega \in I_\delta, D(\omega, z, t)$, can be analytically continued in z from the upper to the lower half-plane to yield a function holomorphic in a connected region containing the point $\lambda = U - i\gamma/2$. In the most favourable case, such a region is the whole z-plane cut along $U - \delta < \operatorname{Re} x < U + \delta$ with $x = \operatorname{Re} z$. In the general case, $D(\omega, z, t)$ shall have other singularities (poles, cuts) in z, depending in general on ω. However, we assume that there are two intervals $U - \Delta_1 < x < U - \Delta_2$, $U + \Delta_2 < x < U + \Delta_1$ with $\Delta_1 > \Delta_2 > \delta$ such that no branch point nor pole of $D(\omega, z, t)$, lie on them for $\omega \in I_\delta$ so that $D(\omega, z, t)$ can be analytically continued for all $\omega \in I_\delta$ along the arrows indicated in Fig. 1.

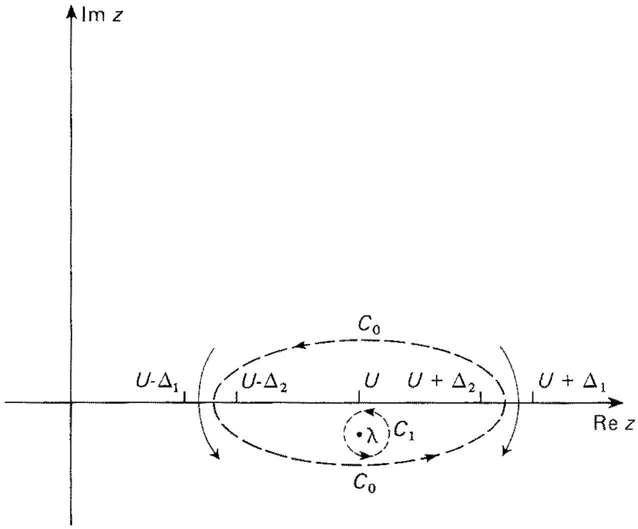

Figure 1.

(b) Consider now the following expression, with $g = \{g(\mathbf{p})u\} \in \mathscr{H}$,

$$f^{(1,2)} = \{f^{(1,2)}(\mathbf{p})\} = \{f(\mathbf{P}, \mathbf{q})v\} \in \mathscr{H}_{12}, \qquad f(\mathbf{P}, \mathbf{q}) = 0 \qquad \text{for } \omega \notin I_\delta$$

$$\frac{1}{2\pi i}\int d\mathbf{P} \exp\left(-i\frac{P^2}{2M}t\right)\oint_C dz \frac{1}{z-\lambda}(\overleftarrow{G}_0^{(1,2)}f^{(1,2)}(\mathbf{P}),$$

$$\exp(-i\bar{E}_0 t)\overleftarrow{G}_{(+)0}(z)g(\mathbf{P})u)_{\bar{h}_0,\,\text{a.c.}} = \frac{1}{2\pi i}\int d\mathbf{P}$$

$$\exp\left(-i\frac{P^2}{2M}t\right)g(\mathbf{P})\int d\mathbf{q} f^*(\mathbf{P},\mathbf{q})\oint_C dz \frac{1}{z-\lambda} D_+(\omega, z, t) \tag{5.7}$$

where $D_+(\omega, z, t)$ is the analytic continuation of $D(\omega, z, t)$ into the lower half-plane. In fact, with the assumption just made one finds that the scalar product in the left-hand side of (5.7) can be actually analytically continued along the arrows in Fig. 1, and the continuation is just given by the right-hand side of (5.7). Obviously the result of the integration on C is quite different according to whether C is of type C_0 or C_1 in Fig. 1. In fact, only in the first case does C contain the pole of $D_+(\omega, z, t)$ at $z = \omega + U_1 + U_2$. The main poin now is that expression (5.7) coincides with

$$\lim_{n\to\infty}(\overleftarrow{G}^{(1,2)}f^{(1,2)}, e^{-i\bar{H}t}\overleftarrow{G}_{(+)n}g)_{\bar{\mathscr{H}}} \tag{5.7$'$}$$

where $G_{(+)n}$ is just the sequence appearing in (4.1$'$) and explicitly built in Appendix 2. Definition (5.7$'$) can be proved exactly by the same technique used in Appendix 2, simply replacing $F(z_1, z_2, t)$ by

$$\widetilde{F}(z, \omega, t) = \frac{1}{2\pi i}\frac{1}{z-\lambda}D_+(\omega, z, t) \tag{5.8}$$

6. *Decay of an Unstable Particle (Continued)*

From (4.2), (4.3) and (5.7$'$) we see that

$$P(g, t_0 \to f^{(1,2)}, t) = \left|\frac{1}{2\pi i}\int d\mathbf{P}\exp\left[-i\frac{P^2}{2M}(t-t_0)\right]\oint dz \frac{1}{z-\lambda}\right.$$

$$\left.\cdot\,(\overleftarrow{G}_0^{(1,2)}f^{(1,2)}(\mathbf{P}), \exp[-i\bar{E}_0(t-t_0)]\,\overleftarrow{G}_{(+)0}(z)g(\mathbf{P})u)_{\bar{h}_0,\,\text{a.c.}}\right|^2 \tag{6.1}$$

and that effects corresponding to elements of $\mathscr{H}_{12}$, such that $f(\mathbf{P}, \mathbf{q}) = 0$ for $\omega \in I_\delta$, are preparation independent: this is quite reasonable from a physical point of view, since it means that the features of the decay which are related to a neighbourhood of the peak in the decay spectrum in $\omega = \omega_0$ are preparation independent.

It remains to decide what of the two types of paths C in Fig. 1 to choose. To investigate this point, we take account of (5.5) in (6.1), so that

$$P(g, t_0 \to f^{(1,2)}, t) = \left| \int d\mathbf{P} \exp\left[-i \frac{P^2}{2M}(t - t_0)\right] g(\mathbf{P}) \int d\mathbf{q} f^*(\mathbf{P}, \mathbf{q}) \right.$$
$$\left\{ \frac{Z^{1/2}}{2\pi i} \oint_C dz \cdot \frac{\exp[-iz(t - t_0)]}{z - \lambda} (\bar{u}^{(1,2)}_{\mathbf{q},\,\mathrm{a.s.}}, [\bar{P}_0 - \bar{N}_0(z)]\bar{u})_{\bar{h}_0,\,\mathrm{a.c.}} \right.$$
$$- \frac{Z^{1/2}}{(2\pi i)^2} \int_{-\infty + i\epsilon}^{+\infty + i\epsilon} dz' \exp[-iz'(t - t_0)] \cdot \oint_C dz \frac{1}{z - \lambda} \frac{1}{z' - z} (\bar{u}^{(1,2)}_{\mathbf{q},\,\mathrm{a.s.}},$$
$$\left.\left. [\bar{P}_0 - \bar{N}_0(z')] \frac{1}{z' + \bar{M}_0(z')} [z + \bar{M}_{0(+)}(z)]\bar{u})_{\bar{h}_0} \right\} \right|^2 \tag{6.2}$$

where $\bar{M}_{0+}(z)$ is the operator $\bar{M}_0(z)$ analytically continued from the upper to the lower half-plane. Recalling that (Lanz *et al.*, 1974)

$$[\bar{M}_{0+}(\lambda) + \lambda]\bar{u} = 0 \tag{6.3}$$

one sees that the second term in brackets in (6.2) disappears, so that

$$P(g, t_0 \to f^{(1,2)}, t) = \left| \int d\mathbf{P} \exp\left[-i \frac{P^2}{2M}(t - t_0)\right] g(\mathbf{P}) \cdot \int d\mathbf{q} f^*(\mathbf{P}, \mathbf{q}) \frac{Z^{1/2}}{2\pi i} \right.$$
$$\left. \cdot \oint dz \frac{\exp[-iz(t - t_0)]}{z - \lambda} (\bar{u}_{\mathbf{q},\mathrm{a.s.}}, [\bar{P}_0 - \bar{N}_0(z)]\bar{u})_{\bar{h}_0,\,\mathrm{a.c.}} \right|^2 \tag{6.4}$$

If we consider the path C_1 in Fig. 1, the only singularity in the integral over z is the pole in $z = \lambda$. The result is then an expression decaying in time as $\exp(-i\gamma t/2)$. This is quite unreasonable from a physical point of view, since it would mean that there are no decay products after a certain time. Therefore we shall choose a path C of the type C_0 in Fig. 1.

In the case that $z = \lambda$ and $z = U_1 + U_2 + \omega$ are the only singularities in (6.4), we obtain

$$P(g, t_0 \to f^{(1,2)}, t) = \left| \int d\mathbf{P} \exp\left(-i \frac{P^2}{2M} t\right) g(\mathbf{P}) \int d\mathbf{q} f^*(\mathbf{P}, \mathbf{q}) Z^{1/2} \right.$$
$$\cdot \left\{ \frac{\exp[-i(U_1 + U_2 + \omega)(t - t_0)]}{U_1 + U_2 + \omega - \lambda} \left[\lim_{z \to U_1 + U_2 + \omega} (z - U_1 - U_2 - \omega) \right.\right.$$
$$\left. \cdot (\bar{u}^{(1,2)}_{\mathbf{q},\,\mathrm{a.s.}}, [\bar{P}_0 - \bar{N}_0(z)]\bar{u})_{\bar{h}_0,\,\mathrm{a.c.}} \right] - \frac{\exp[-i\lambda(t - t_0)]}{U_1 + U_2 + \omega - \lambda}$$
$$\left.\left. \cdot \left[\lim_{z \to \lambda} (z - U_1 - U_2 - \omega) \cdot (\bar{u}^{(1,2)}_{\mathbf{q},\,\mathrm{a.s.}}, [\bar{P}_0 - \bar{N}_0(z)]\bar{u})_{\bar{h},\,\mathrm{a.c.}} \right] \right\} \right|^2 \tag{6.5}$$

One might be surprised that in general $P(g, t_0 \to f^{(1,2)}, t) \neq 0$ for $t = t_0$. However (6.5) can be considered as a true transition probability only if $f^{(1,2)}$ describes well-separated particles, in which case P practically vanishes until the decay products are at a macroscopic distance from each other. We give finally the probability for the production of the unstable particle

$$P(f^{(1,2)}, t_0 \to g, t) = \left| \frac{1}{2\pi i} \int d\mathbf{P}\, exp\left[-i\frac{P^2}{2M}(t - t_0)\right] \oint_C dz\, \frac{1}{z - \lambda} \right.$$
$$\left. \cdot\, (\overleftarrow{\bar{G}}_{(-)0}(z) g(\mathbf{P})\bar{u}, \exp[-i\bar{E}_0(t - t_0)]\, \overleftarrow{\bar{G}}_0^{(12)} f^{(12)}(\mathbf{P}, \mathbf{q}))_{\bar{h}_0, \mathrm{a.c.}} \right|^2 \tag{6.6}$$

7. *Examples*

Let us consider some simple models, in which one can study explicitly the singularities of $\bar{N}_0(z)$, $\bar{M}_0(z)$, etc. The problem of the elimination of the cut-off from the model field theory is outside the scope of the present paper.

(a) 'Galilee' model (Levy-Leblond, 1967). Let

$$\bar{H} = \bar{H}_F + \bar{H}_I \tag{7.1}$$

$$\bar{H}_F = \int d\mathbf{p} \left(\frac{p^2}{2\mathcal{M}} + U_0\right) \psi_V^+(\mathbf{p}) \psi_V(\mathbf{p}) + \int d\mathbf{p}'\, \frac{p'^2}{2M}\, \psi_N^+(\mathbf{p}') \psi_N(\mathbf{p}')$$
$$+ \int d\mathbf{K}\, \frac{\mathbf{K}^2}{2m}\, a^+(\mathbf{K}) a(\mathbf{K})$$

$$\bar{H}_I = \lambda_0 \int d\mathbf{p}\, d\mathbf{q} f(\omega) \left[\psi_V^+(\mathbf{p}) \psi_N\left(\frac{M}{\mathcal{M}}\mathbf{p} + \mathbf{q}\right) a\left(\frac{m}{\mathcal{M}}\mathbf{p} - \mathbf{q}\right) + \mathrm{h.c.}\right]$$

$$\mathcal{M} = M + m, \qquad \mathcal{M}\mu = mM, \qquad \omega = \frac{q^2}{2\mu}.$$

For further details on the symbols, we refer to Levy-Leblond (1967), or to B. $\bar{P}_0$ is chosen as the projector onto the eigenspace of the operator $\bar{E}_{0F}$ with eigenvalue U_0, where $\bar{E}_{0F}$ is the internal energy operator for the bare particles In B we deduced the equation giving internal energy and life-time for the unstable V particles. To describe the decay products we consider (6.4), taking into account that in this case (writing $(N\theta)$ instead of (12) and recalling that $\bar{u}^{(N\theta)}_{\mathbf{q},\mathrm{a.s.}} = \bar{u}^{(N\theta)}_{\mathbf{q},F}$):

$$(\bar{u}^{(N\theta)}_{\mathbf{q},F}, [\bar{P}_0 - \bar{N}_0(z)]\, \bar{u})_{\bar{h}_0} = -\lambda_0 f(\omega) \frac{1}{\omega - z} \tag{7.2}$$

Replacing (7.2) into (6.5), where we write $e(\mathbf{P}, \mathbf{q})$ instead of $f(\mathbf{P}, \mathbf{q})$ to avoid confusion with the cut-off function $f(\omega)$ in (7.1), we obtain by easy calculations:

$$P(g, t_0 \to e^{(N\theta)}, t) = \left| \int d\mathbf{P} \exp\left[-i \frac{P^2}{2\mathcal{M}}(t - t_0)\right] g(\mathbf{P}) \right.$$

$$\left. \times \int d\mathbf{q}\, e^*(\mathbf{P}, \mathbf{q}) Z^{1/2} \lambda_0 f(\omega) \frac{\exp[-i\omega(t - t_0)] - \exp[-i\lambda(t - t_0)]}{\omega - \lambda} \right|^2 \tag{7.3}$$

We have therefore obtained, without approximations, a result which, in the framework of the usual damping theory, is reached through rather strong approximations: this difference is due to the introduction of the 'generalised dressed states' (4.5′), which eliminate the 'approximations' needed to pass from the field to the particle level.

Choosing $e(\mathbf{P}, \mathbf{q}) = e'(\mathbf{P})\delta(\mathbf{q} - \bar{\mathbf{q}})$ and taking the limit $t \to +\infty$ in (7.3), we define $W(\bar{\mathbf{q}})$ as follows

$$P(g, t_0 \to e^{(N\theta)}, +\infty) = t \to \infty \left| \int d\mathbf{P} \exp\left[-i \frac{P^2}{2\mathcal{M}}(t - t_0)\right] g(\mathbf{p})\, e'^*(\mathbf{P}) \right|^2 W(\bar{\mathbf{q}})$$

The decay spectrum $W(\omega)$ is defined by

$$W(\omega)\, d\omega = \int_\Omega W(\bar{\mathbf{q}}) \bar{q}^2\, d\,|\bar{\mathbf{q}}|\, d\Omega$$

so that by (7.3)

$$W(\omega) = 4\pi(2\mu^3\omega)^{1/2} \lambda_0{}^2 f^2(\omega) \frac{|Z|}{(\omega - U)^2 + \gamma^2/4} \tag{7.4}$$

A particularly clear interpretation is obtained in the weak coupling limit, in which the life-time of the unstable particle becomes long: in fact, long-lived particles are the only ones which we can observe directly by means of, for example, a bubble chamber. When $\lambda_0 \ll 1$ also $(|Z| - 1) \ll 1$ and $\gamma \ll 1$; from equation (5.2) of B we see that for $\gamma \ll 1$ one has practically

$$\frac{\gamma}{2} = 4\pi^2 \lambda_0{}^2 \sqrt{(2\mu^3 U)} f^2(U) \tag{7.5}$$

since

$$\frac{1}{\pi} \frac{\gamma/2}{(\omega - U)^2 + \gamma^2/4} \xrightarrow{\gamma \to 0} \delta(\omega - U) \tag{7.6}$$

we obtain from (7.4) a perfect Breit–Wigner decay spectrum:

$$W(\omega) = \frac{1}{\pi} \frac{\gamma/2}{(\omega - U)^2 + \gamma^2/4} \tag{7.7}$$

In the weak coupling limit one also obtains quite a pictorial description of the decay process. To see this, let us choose the localised states:

$$e(\mathbf{P}, \mathbf{q}) = \frac{1}{(2\pi)^3} e^{-i\mathbf{P}\cdot\mathbf{x}_b} e^{-i\mathbf{q}\cdot\mathbf{x}_r} \tag{7.8}$$

$$\mathbf{x}_b = \frac{M\boldsymbol{\varkappa}' + m\boldsymbol{\varkappa}}{\mathscr{M}}, \qquad \mathbf{x}_r = \mathbf{x}' - \mathbf{x}$$

which could ideally correspond to an apparatus revealing N particles localised in $\boldsymbol{\varkappa}'$ and θ particles localised in $\boldsymbol{\varkappa}$; $\boldsymbol{\varkappa}$ and $\boldsymbol{\varkappa}'$ are chosen well separated from each other. Then the expression in the modulus in the right-hand side of (7.3) becomes, by quite easy calculations,

$$\frac{1}{(2\pi)^{1/2}} \mu\bar{g}(\mathbf{x}_b, t-t_0) \frac{\lambda_0}{i\,|\mathbf{x}_r|} \int_0^\infty d\omega \frac{f(\omega)}{\omega - \lambda}$$
$$\times \{e^{i\sqrt{(2\mu\omega)}|\mathbf{x}_r|} - e^{-i\sqrt{(2\mu\omega)}|\mathbf{x}_r|}\}\{e^{-i\omega(t-t_0)} - e^{-i\lambda(t-t_0)}\} \tag{7.9}$$

with

$$\bar{g}(\mathbf{x}_b, t-t_0) = \frac{1}{(2\pi)^{3/2}} \int d\mathbf{P} g(\mathbf{P}) \exp\left\{i\left[\mathbf{P}\cdot\mathbf{x}_b - \frac{P^2}{2\mathscr{M}}(t-t_0)\right]\right\} \tag{7.10}$$

Taking into account that for $\lambda_0 \ll 1$ the factor $(\omega - \lambda)^{-1}$ is strongly peaked around $\bar{\omega} = U$, we can take all the factors except the exponentials outside the integral sign, extend the integral from $-\infty$ to $+\infty$ and develop $\sqrt{(2\mu\omega)} \simeq \sqrt{(2\mu U)} + (\omega - U)\bar{V}^{-1}$; with $\bar{V} = \sqrt{(2U\mu^{-1})}$, thereby neglecting the spreading of wave packets. Then the integral becomes elementary and one obtains:

$$(2\pi)^{1/2}\mu\bar{g}(\mathbf{x}_b, t-t_0) \frac{\lambda_0 f(U)}{|\mathbf{x}_r|} \exp[i\{\sqrt{(2\mu U)}|\mathbf{x}_r| - U(t-t_0)\}$$
$$\cdot \exp\left\{-\frac{\gamma}{2}\left[t - t_0 - \frac{|\mathbf{x}_r|}{\bar{V}}\right]\right\}\theta\left[t - t_0 - \frac{|\mathbf{x}_r|}{\bar{V}}\right] \tag{7.11}$$

Then finally with (7.9)

$$P(g, t_0 \to e^{(N\theta)}, t) = 2\pi\mu^2 |\bar{g}(\mathbf{x}_b, t-t_0)|^2 \frac{\lambda_0^2 f^2(U)}{|\mathbf{x}_r|^2}$$
$$\cdot \exp\left\{-\gamma\left[t - t_0 - \frac{|\mathbf{x}_r|}{\bar{V}}\right]\right\}\theta\left[t - t_0 - \frac{|\mathbf{x}_r|}{\bar{V}}\right] \tag{7.12}$$

Equation (7.12) gives a pictorial description of the decay and justifies the experimental verifications of the exponential decay low in which the decay products are observed. In fact, if one keeps the apparatuses in $\mathbf{x}$ and $\mathbf{x}'$ fixed the time plot of their response is exponential by (7.11); further, if one counts the decay products in all space, one finds by (7.11) the result $(1 - e^{-\gamma(t-t_0)})\cdot N$, where N is the number of unstable V particles at time $t = t_0$, so that the decay frequency at time t $N.(d/dt)(1 - e^{-\gamma(t-t_0)})$ is likewise exponential.

Expression (7.10) can be even considered as a 'wave function' $\psi(\mathbf{x}_b, \mathbf{x}_r, t - t_0) \in L^2(R_6)$ for the $N - \theta$ system; together with the 'wave function'

$$\psi(\mathbf{x}_b, t - t_0) = \bar{g}(\mathbf{x}_b, t - t_0) \exp\left[-\frac{\gamma}{2}(t - t_0)\right] \in L^2(R_3)$$

it obeys the unitarity relation $\|\psi(t)\|^2_{L^2(R_6)} + \|\varphi(t)\|^2_{L^2(R_3)} = 1$. We stress the presence of the factor $\theta[t - t_0 - (|\mathbf{x}_r|/\bar{V})]$, which makes the wave function ψ normalisable, despite the divergent factor

$$\exp\left[-\frac{\gamma}{2}\left(t - t_0 - \frac{|\mathbf{x}_r|}{\bar{V}}\right)\right]$$

(Rosenfeld, 1972).

From (7.3) we can easily obtain an 'S-matrix' amplitude for the transition $V \to N\theta$. To this extent, we proceed as in (3.8), replacing in (7.3) $e(\mathbf{P}, \mathbf{q})$ by $e(\mathbf{P}, \mathbf{q}) \exp[-i(P^2/2\mathcal{M})t - i\omega t]$ and $g(\mathbf{p})$ by $g(\mathbf{p}) \exp\{-iUt_0 - i(P^2/2\mathcal{M})t_0\}$, which corresponds to take the elements $f^{(t)}$ and $g^{(t_0)}$ defined in (3.7). Physically, we should consider times t_0 and t such that $t - t_0$ is very much larger than the $N\theta$ collision time (so that the decay products are separated), but still much shorter than the life-time, which is long enough. We do this making the double limit $t \to +\infty$, $\gamma \to 0$, in such a way that $\gamma t \to 0$. We then obtain from (7.3), by the usual relation

$$\lim_{l \to \infty} \frac{1}{\pi} \frac{\sin l\mathbf{x}}{\mathbf{x}} = \delta(\mathbf{x}) \tag{7.13}$$

that

$$P = \left| \int d\mathbf{P} \exp\left(-i\frac{P^2}{2\mathcal{M}}t\right) g(\mathbf{P}) \int d\mathbf{q}\, e^*(\mathbf{P}, \mathbf{q}) \{-2\pi i \delta(\omega - U) . Z^{1/2} \lambda_0 f(\omega) \right|^2 \tag{7.14}$$

The amplitude $Z^{1/2}\lambda_0 f(U) \sim \lambda_0 f(U)$ coincides with the one defined in the framework of S-matrix theory (Eden *et al.*, 1966; Zwanziger, 1963), as one of the two factors which factorise the residue of the amplitude for the collision of the decay products, analytically continued in the squared centre of mass energy (in our Galilean case, in the internal energy ω) to reach the pole corresponding to the unstable particle (in our case, $\omega = \lambda$) on an unphysical sheet. When (6.5) holds, one easily sees that an amplitude for the transition from the unstable particle to its decay products can be obtained by a trivial extension of the procedure used to derive (7.14).

Finally we give the probability for the production of a V particle, obtained from (6.6):

$$P(e^{(N\theta)}, t_0 \to g, t) = \left| \int d\mathbf{P} \exp\left[-i \frac{P^2}{2\mathscr{M}}(t - t_0)\right] g^*(\mathbf{P}) \right.$$

$$\left. \cdot \int d\mathbf{q}\, e(\mathbf{P}, \mathbf{q}) Z^{1/2} \lambda_0 f(\omega) \frac{e^{-i\omega(t-t_0)} - e^{-i\lambda(t-t_0)}}{\omega - \lambda} \right|^2 \tag{7.15}$$

Of course, (7.15) is not the 'time inverted' (7.3), since it exhibits a decaying exponential–not an increasing one–as well as (7.3). By a suitable choice of $g(\mathbf{P})$ and $e(\mathbf{P}, \mathbf{q})$ and by explicitly evaluating (7.15) one obtains a pictorial description of the formation of the V particle in a $N - \theta$ collision. The probability of such a process is appreciable only if the two wave packets in the initial state are such that they will overlap during their free motion.

We consider now two models slightly more complicated than the 'Galilee', which provide operators $\bar{M}_0(z)$ and $\bar{N}_0(z)$ with more involved analyticity properties.

(b) Let us consider the Hamiltonian

$$\bar{H} = \bar{H}_F + \bar{H}_I + \bar{H}_{II} \tag{7.16}$$

where $\bar{H}_F$ and $\bar{H}_I$ are given by (7.1) and

$$\bar{H}_{II} = \lambda_1 \int d\mathbf{P}\, d\mathbf{q}_1\, d\mathbf{q}_2 v(\omega_1, \omega_2) N^+\left(\frac{M}{\mathscr{M}}\mathbf{P} + q_1\right)$$

$$\cdot a^+\left(\frac{m}{\mathscr{M}}\mathbf{P} - \mathbf{q}_1\right) N\left(\frac{M}{\mathscr{M}}\mathbf{P} + \mathbf{q}_2\right) a\left(\frac{m}{\mathscr{M}}\mathbf{P} - \mathbf{q}_2\right) \tag{7.16'}$$

where $v(\omega_1, \omega_2)$ is a real function, which we assume to depend analytically on its arguments. In this model we can give only perturbative expressions for $\bar{M}_0(z)$ and $\bar{N}_0(z)$ in the coupling λ_1. The projector $\bar{P}_0$ is the same used in the Galilee model. We have:

$$\bar{M}_0(z) = \left\{ -U_0 + \lambda_0^2 \left[\int d\mathbf{q}' \frac{f^2(\omega')}{\omega' - z} + \lambda_1 \int \frac{f(\omega'')}{\omega'' - z} \right.\right.$$

$$\left.\left. \times\ v(\omega'', \omega') \cdot \frac{f(\omega')}{\omega' - z}\, d\mathbf{q}'\, d\mathbf{q}'' + \dots \right] \right\} \bar{P}_0 \tag{7.17}$$

$$(\bar{u}^{(N\theta)}_{\mathbf{q},F}, [\bar{P}_0 - \bar{N}_0(z)]\bar{u})_{\bar{h}} = -\lambda_0 \frac{1}{\omega - z}\left[f(\omega) - \lambda_1 \int d\mathbf{q}' v(\omega, \omega') \frac{f(\omega')}{\omega' - z} \right.$$

$$\left. + \lambda_1^2 \int d\mathbf{q}'\, d\mathbf{q}'' v(\omega, \omega'') \cdot \frac{1}{\omega'' - z} \cdot v(\omega'', \omega') \frac{f(\omega')}{\omega' - z} + \dots \right] \tag{7.18}$$

Then (7.18) has, in addition to the usual pole in $z = \omega$, a cut from 0 to $+\infty$; (7.18) can be analytically continued in the lower half-plane since $v(\omega_1, \omega_2)$ is analytic. Finally, if one calculates the transition amplitude one obtains, apart a momentum-conserving δ-function, the result $(\bar{u}^{(N\theta)}_{\mathbf{P},\mathbf{q},\text{out}}, H_I \bar{u}_{\mathbf{P},F})_{\bar{\mathscr{H}}}$ where $\bar{u}_{\mathbf{P},F} \in \mathscr{H}$ is the bare V-particle state and $\bar{u}^{(N\theta)}_{\mathbf{P},\mathbf{q},\text{out}}$ is the outgoing state of the $N-\theta$ system, calculated for $\lambda_0 = 0$.

(c) Let us consider a model in which five elementary quanta V, V', N, θ, θ' interact according to the scheme $V \leftrightarrow V' + \theta$, $V' \leftrightarrow N + \theta'$. Then

$$\bar{H} = \bar{H}_F' + \bar{H}_I + \bar{H}_{III} \tag{7.19}$$

where $\bar{H}_I$ is given by (7.1) with ψ_N replaced by $\psi_{V'}$, and

$$\begin{aligned} \bar{H}_F' = \int d\mathbf{p}\left(\frac{p^2}{2\mathscr{M}} + U_0\right)\psi_V^+(\mathbf{p})\psi_V(\mathbf{p}) \\ + \int d\mathbf{p}'\left(\frac{p'^2}{2M'} + U_0'\right)\psi_{V'}^+(\mathbf{p}')\psi_{V'}(\mathbf{p}') + \int d\mathbf{q}\,\frac{q^2}{2M}\,\psi_N^+(\mathbf{q})\psi_N(\mathbf{q}) \\ + \int d\mathbf{k}\,\frac{\mathbf{k}^2}{2m}\,a^+(\mathbf{k})a(\mathbf{k}) + \int d\mathbf{k}'\,\frac{\mathbf{k}'^2}{2m'}\,a'^+(\mathbf{k}')a'(\mathbf{k}') \end{aligned} \tag{7.19'}$$

$$\bar{H}_{III} = \lambda' \int d\mathbf{P}\,d\mathbf{q}' f'(\omega')\left[\psi_{V'}^+(\mathbf{P})\psi_N\left(\frac{M}{M'}\,\mathbf{P} + \mathbf{q}'\right).a'\left(\frac{m'}{M'}\,\mathbf{P} - \mathbf{q}'\right) + \text{h.c.}\right]$$

$$\omega' = \frac{q'^2}{2\mu'}, \qquad \mu' M' = m' M, \tag{7.19''}$$

where $f'(\omega)$ is a real cut-off function depending analytically on ω, $\psi_{V'}(\mathbf{p})$ and $a'(\mathbf{k})$ are fields analogous to $\psi_V(\mathbf{p})$ and $a(\mathbf{k})$ with masses M' and m' respectively, such that $M + m' = M'$, $M' + m = \mathscr{M}$.

We consider, aside from the usual projector $\bar{P}_0$ for the V particles, a projector $\bar{P}_0'$ for the V' particles. $\bar{P}_0'$ is defined as the projector onto the eigenspace of the operator $\bar{E}_{0F}'$ with eigenvalue U_0'. We have correspondingly two operators $\bar{M}_0'(z)$, $\bar{M}_0(z)$ given by:

$$\begin{aligned} \bar{M}_0'(z) = \mathscr{M}_0'(z)\bar{P}_0' \\ \mathscr{M}_0'(z) = -U_0' + \lambda' \int d\mathbf{q}' f'^2(\omega')\,\frac{1}{\omega' - z} \end{aligned} \tag{7.20}$$

$$\bar{M}_0(z) = \mathscr{M}_0(z)\bar{P}_0$$

$$\mathscr{M}_0(z) = -U_0 + \lambda_0 \int d\mathbf{q}'' f^2(\omega'')\,\frac{1}{U_0' + \omega'' - z - \lambda'^2 \int d\mathbf{q}' f'^2(\omega')\,\dfrac{1}{\omega' + \omega'' - z}}$$

$$\omega' = \frac{q'^2}{2\mu'}, \qquad \omega'' = \frac{q''^2}{2\mu}, \qquad \mathscr{M}\mu = mM' \tag{7.21}$$

We assume that the parameters are chosen in such a way that we can embed into the model a stable V' particle and an unstable V particle. Therefore we consider a real solution $z = U' < 0$ of equation $z + \mathscr{M}'_0(z) = 0$, and a complex solution $z = U - i\gamma/2$ of equation $z + \mathscr{M}_{0+}(z) = 0$.

Since from (7.20) $\bar{M}_0(z)$ has a cut from U' to $+\infty$, we have $U > U'$. To describe the decay $V \to V' + \theta$, we must consider the expression

$$(u^{(V'\theta)}_{\mathbf{q},\mathrm{a.s.}}, [\bar{P}_0 - \bar{N}_0(z)]\bar{u})_{\bar{h}_0} = -\lambda_0 f(\omega) \cdot \frac{1}{U'_0 + \omega - z - \lambda'^2 \int d\mathbf{q}' f'^2(\omega') \dfrac{1}{\omega' + \omega - z}}$$

$$\times \left\{ 1 + \frac{1}{z - U' - \omega} \cdot \left[\lambda'^2 \int d\mathbf{q}' \frac{f'^2(\omega')}{\omega' + \omega - z} - \lambda'^2 \int d\mathbf{q}' \frac{f'^2(\omega')}{\omega' - U'} \right] \right\}$$

$$\omega = \frac{q^2}{2\mu}, \qquad \omega' = \frac{q'^2}{2\mu} \tag{7.22}$$

(7.22) has a pole for $z = U' + \omega$, and a cut from ω to $+\infty$. This cut depends on ω, which did not happen in case (b). However, since $U > U'$, restricting the values of ω to a suitable neighbourhood of $\omega_0 = U - U'$, such that $\omega_{\max} - \omega_{\min} < -U'$, one gets that such a cut remains outside the path C. Therefore assumptions of Section 5 are satisfied.

It is interesting to calculate also the transition probability $V \to N\theta'\theta$, in the case $U > 0$. Such a probability is given by a straightforward generalisation of (6.1); the relevant matrix element is

$$(u^{(N\theta'\theta)}_{\mathbf{q},\mathbf{q}',F}, [\bar{P}_0 - \bar{N}_0(z)]\bar{u})_{\bar{h}} \tag{7.23}$$

where $\mathbf{q}$ and $\mathbf{q}'$ are the relative Jacobi coordinates for the three-particle system $N\theta'\theta$; specifically, $\mathbf{q}'$ is the relative momentum of the two particles N and θ'. One has immediately that (7.23) is given by

$$\frac{1}{U'_0 + \omega - z - \lambda'^2 \int d\mathbf{q}'' f'(\omega'') \dfrac{1}{\omega'' + \omega - z}} \; \frac{1}{\omega + \omega' - z} \; \lambda_0 f(\omega)$$

$$\omega = \frac{q^2}{2\mu}, \qquad \omega' = \frac{q'^2}{2\mu'}, \qquad \omega'' = \frac{q''^2}{2\mu'} \tag{7.23'}$$

Therefore (7.23) has not only the expected pole in $z = \omega + \omega'$, but also a pole in $z = \omega + U' < \omega$ and a branching point at $z = \omega$. However, by suitably restricting ω and ω' to values such that $\omega + \omega'$ is near U, ω' is not too small and $\omega_{\max} < \omega_{\min} + \omega'_{\min}$, one gets that such singularities remaining outside C. On the contrary, in the case where also the V' particle is unstable, the pole $z = \omega + U' - i\gamma^1/2$ (in this case $U' > 0$) would lie inside C and give a relevant contribution, since as one easily sees the process $V \to N\theta'\theta$ in this case proceeds mainly in two steps, namely $V \to V'\theta \to N\theta'\theta$. We stress that in the usual damping theory, in which one does not select preparation-independent effects, one would take into account the contribution of the pole $z = \omega + U'$, even in the case where V' is stable, which appears incorrect.

8. *Final Remarks*

It seems to us that the present analysis of the decay process is, in many respects, more complete and satisfactory than that of the damping theory, and that the present treatment is less *ad hoc* than other approaches (see, for example, Nazanishi (1958), or the treatment of metastable states in Hugenholtz (1957)).

It is remarkable that in the limit of long life-time, in which one has energy-conserving transition probabilities, one gets a pictorial description of the decay process and a quite natural extension of the S-matrix to the processes involving unstable particles in the initial and/or final configurations. This point will be further illustrated and discussed in Casagrande & Lugiato (1973).

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Appendix 1

The Asymptotically Stationary States

In the Ket notation, the stable particle O_1 is described in the model by the orthonormal set of dressed states $\bar{u}^{(1)}_{\mathbf{p}_1}$, $\mathbf{p}_1 \in R_3$ such that

$$\bar{H}\bar{u}^{(1)}_{\mathbf{p}_1} = \left(U_1 + \frac{P_1{}^2}{2M_1}\right)\bar{u}^{(1)}_{\mathbf{p}_1} \tag{A.1.1}$$

U_1 being the internal energy of O_1; one can write

$$\bar{u}^{(1)}_{\mathbf{p}_1} = A^{+}_{1;\mathbf{p}_1}\bar{u}_0 \tag{A.1.2}$$

where $\bar{u}_0$ is the physical vacuum:

$$\bar{H}\bar{u}_0 = 0 \tag{A.1.3}$$

and $A^{+}_{1;\mathbf{p}_1}$ is a suitable linear combination, with real coefficients, of products of creation operators of the bare quanta of the fields, defined in Van Hove (1955, 1956) and Hugenholtz (1957). The coefficients of the linear combination are essentially residues of suitable matrix elements of the various terms in the perturbation expansion of the resolvent $[\bar{H} - z]^{-1} = [\bar{H}_F + \bar{H}_I - z]^{-1}$ in the interaction H_I, calculated in $z = U_1 + P_1{}^2/2M_1$.

Under space translations, rotations and accelerations $A^+_{1;\mathbf{p}_1}$ transform exactly as the creation operators $a^{+(\text{in})}_{1;\mathbf{p}_1}, a^{+(\text{out})}_{1;\mathbf{p}_1}$ which, when applied successively to $\bar{u}_0$, build the orthonormal Kets describing 1, 2 . . . incoming or outgoing O_1 particles respectively. On the contrary, the transformation of $A^+_{1;\mathbf{p}_1}$ under time translations is different from that of $a^{+(\text{in, out})}_{1;\mathbf{p}_1}$; further, the $A^+_{1;\mathbf{p}_1}$ do not obey canonical commutation relations with their adjoints.

One has likewise for the stable particle O_2 the orthonormal set $\bar{u}^{(2)}_{\mathbf{p}_2}$, with

$$\bar{u}^{(2)}_{\mathbf{p}_2} = A^+_{2;\mathbf{p}_2}\bar{u}_0 \tag{A.1.2$'$}$$

Let us assume for simplicity that O_1 and O_2 are the only stable particles contained in the model. Let us consider the Kets

$$\begin{gathered}
\bar{u}^{(1)}_{\mathbf{p}_1,\text{a.s.}} = \bar{u}^{(1)}_{\mathbf{p}_1}, \qquad \bar{u}^{(2)}_{\mathbf{p}_2,\text{a.s.}} = \bar{u}_{\mathbf{p}_2} \\
\bar{u}^{(11)}_{\mathbf{p}_1\mathbf{p}_1',\text{a.s.}} = A^+_{1;\mathbf{p}_1}A^+_{1;\mathbf{p}_1'}\bar{u}_0, \qquad \bar{u}^{(22)}_{\mathbf{p}_2,\mathbf{p}_2',\text{a.s.}} = A^+_{2;\mathbf{p}_2}A^+_{2;\mathbf{p}_2'}\bar{u}_0 \\
\bar{u}^{(12)}_{\mathbf{p}_1\mathbf{p}_2,\text{a.s.}} = A^+_{1;\mathbf{p}_1}A^+_{2;\mathbf{p}_2}\bar{u}_0 \\
\bar{u}^{(111)}_{\mathbf{p}_1,\mathbf{p}_1',\mathbf{p}_1'',\text{a.s.}} = A^+_{1;\mathbf{p}_1}A^+_{1;\mathbf{p}_1'}A^+_{1;\mathbf{p}_1''}\bar{u}_0, \text{ etc.}
\end{gathered} \tag{A.1.4}$$

These non-orthogonal Kets, obtained applying successively the operators $A^+_{1;\mathbf{p}_1}, A^+_{2;\mathbf{p}_2}$ to $\bar{u}_0$, have the fundamental property of being 'asymptotically stationary', meaning what follows: indicating complexively by α the set of momenta specifying each Ket (A.1.4), if

$$\bar{f}_t = \int d\alpha f(\alpha)\, e^{-iE_\alpha t}\bar{u}_{\alpha,\text{a.s.}} \tag{A.1.5}$$

where E_α is the sum of the energies of the particles in $\bar{u}_{\alpha,\text{a.s.}}$ then one has

$$\lim_{t\to\pm\infty} \| \bar{f}_{t+\tau} - e^{-i\bar{H}\tau}\bar{f}_t \|_{\bar{\mathscr{H}}} = 0 \tag{A.1.6}$$

Due to such a property, the a.s.s. are the analogue of the unperturbed states in potential scattering theory; suitable superpositions of a.s.s. corresponding to well-separated wave packets describe dressed particles moving independently of each other. The a.s.s. have these further properties:

(1) Let us indicate by $\bar{u}^{(\text{in})}_\alpha, \bar{u}^{(\text{out})}_\alpha$ the incoming and outgoing states of the model. The a.s.s. coincide asymptotically to the incoming states for $t \to -\infty$ and to the outgoing states for $t \to +\infty$:

$$\begin{aligned}
&\lim_{t\to-\infty} \| \int d\alpha f(\alpha)\, e^{-iE_\alpha t}[\bar{u}_{\alpha,\text{a.s.}} - \bar{u}^{(\text{in})}_\alpha] \|_{\bar{\mathscr{H}}} = 0 \\
&\lim_{t\to+\infty} \| \int d\alpha f(\alpha)\, e^{-iE_\alpha t}[\bar{u}_{\alpha,\text{a.s.}} - \bar{u}^{(\text{out})}_\alpha] \|_{\bar{\mathscr{H}}} = 0
\end{aligned} \tag{A.1.7}$$

(2) The a.s.s. are asymptotically orthonormal, i.e.

$$\lim_{t\to\pm\infty} \int d\alpha'\, d\alpha\, e^{-i(E_\alpha - E_{\alpha'})t} f^*(\alpha) g(\alpha) (\bar{u}_{\alpha',\text{a.s.}}, \bar{u}_{\alpha,\text{a.s.}}) = \int d\alpha\, f^*(\alpha) g(\alpha) \tag{A.1.8}$$

We stress finally that to the validity of (3.6), and therefore of (3.5), it is essential that $A^+_{1;\mathbf{p}_1}, A^+_{2;\mathbf{p}_2}$ are linear combinations of products of bare creation operators with *real* coefficients.

Appendix 2

Construction of Sequences $\{\overleftarrow{G}_{(+)n}\}, \{\overleftarrow{G}_{(-)n}\}$ *and Proof of Uniform Convergence of* (4.1) *with Respect to* t, f, g

Let us consider the Hilbert spaces $h_1, h_2, \bar{h}$ and two linear operators $\overleftarrow{G}_1(z)$ and $\overleftarrow{G}_2(z)$ bounded holomorphic for $\operatorname{Im} z > 0$, respectively in h_1 into $\bar{h}$ and in h_2 into $\bar{h}$.

Consider the scalar product

$$F(z_1, z_2) = (\overleftarrow{G}_1(z_1^*) f, A\overleftarrow{G}_2(z_2) g) \tag{A.2.1}$$

A being a bounded operator on $\bar{h}$, $f \in h_1, g \in h_2$. Let us assume that the function (A.2.1) of the two complex variables z_1, z_2 can be analytically continued from the upper to the lower half-plane into a generally not simply connected domain D of the product of the two planes of the complex variables z_1, z_2. Let us assume more precisely that a domain D can be found such that (A.2.1) can be continued in D for all f, g with $f \in V_1, g \in V_2$, V_1 and V_2 being two linear manifolds of h_1 and h_2 respectively.

Consider then the expression

$$\int_{C_1} dz_1 \int_{C_2} dz_2 (\overleftarrow{G}_1(z_1^*) f, A\overleftarrow{G}_2(z_2) g)_{\bar{h},\text{a.c.}} \tag{A.2.2}$$

with $C_1 \times C_2$ being a finite path inside D. Then two sequences of linear bounded operators $\{\overleftarrow{G}_{1(-)N}\}, \{\overleftarrow{G}_{2(+)N}\}$ can be found such that

$$\int_{C_1} dz_1 \int_{C_2} dz_2 (\overleftarrow{G}_1(z_1^*) f, A\overleftarrow{G}_2(z_2) g)_{\bar{h},\text{a.c.}}$$
$$= \lim_{\substack{N\to+\infty \\ N'\to+\infty}} (\overleftarrow{G}_{1(-)N} f, A\overleftarrow{G}_{2(+)N'} g)_{\bar{h}}, \qquad \forall\, f \in V_1, g \in V_2 \tag{A.2.3}$$

In fact by the definition of Riemann integral one has

$$\int_{C_1} dz_1 \int_{C_2} dz_2 (\overleftarrow{G}_1(z_1^*) f, A\overleftarrow{G}_2(z_2) g)_{\bar{h},\text{a.c.}}$$
$$= \lim_{\substack{N\to+\infty \\ N'\to+\infty}} \sum_{\kappa=1}^{N} \sum_{\kappa'=1}^{N'} F(\eta_\kappa^{(N)}, \xi_{\kappa'}^{(N')}) \Delta\eta_\kappa^{(N)} \Delta\xi_{\kappa'}^{(N')}$$
$$\Delta\eta_\kappa^{(N)} = \eta_{\kappa+1}^{(N)} - \eta_\kappa^{(N)}, \qquad \Delta\xi_\kappa^{(N')} = \xi_{\kappa+1}^{(N')} - \xi_\kappa^{(N')} \tag{A.2.4}$$

where $\eta_1^{(N)}, \eta_2^{(N)}, \ldots$ are N points on C_1 which give a partition of C_1 into intervals $\Delta\eta_\kappa^{(N)}$ and analogously $\xi_1^{(N')}, \xi_2^{(N')}, \ldots$ give a partition of C_2 into intervals $\Delta\xi_{\kappa'}^{(N')}$.

Let us consider $l+1$ sets of N points $\eta_{i,1}^{(N)}, \eta_{i,2}^{(N)} \ldots \eta_{i,N}^{(N)}$ $(i = 0, 1, \ldots, l)$ with $\operatorname{Im}\eta_{0,j}^{(N)} > 0$ $(j = 1, 2, \ldots N)$, $\eta_{l,j}^{(N)} = \eta_j^{(N)}$ $(j = 1, \ldots, N)$, and analogously $l'+1$ sets of N' points $\xi_{i,1}^{(N')}, \xi_{i,2}^{(N')}, \ldots \xi_{i,N'}^{(N')}$ $(i = 0, 1, \ldots, l')$ with $\operatorname{Im}\xi_{0,j}^{(N')} > 0$ $(j = 1, 2, \ldots N')$, $\xi_{l',j}^{(N')} = \xi_j^{(N')}$ $(j = 1, \ldots, N')$, such that for all $i = 0, 1, \ldots l$, $i' = 0, 1, \ldots, l'$, $\kappa = 1, 2, \ldots, N$, $\kappa' = 1, 2, \ldots, N'$, the Taylor series expansion of $F(z_1, z_2)$ around $z_1 = \eta_{i-1,\kappa}^{(N)}$, $z_2 = \xi_{i'-1,\kappa}^{(N')}$ converges in $z_1 = \eta_{i\kappa}^{(N)}$, $z_2 = \xi_{i\kappa}^{(N')}$, i.e.

$$F(\eta_{i,\kappa}^{(N)}, \xi_{i',\kappa'}^{(N')}) = \sum_{r,s=0}^{\infty} \frac{(\eta_{i,\kappa}^{(N)} - \eta_{i-1,\kappa}^{(N)})^r}{r!} \cdot \frac{(\xi_{i',\kappa'}^{(N')} - \xi_{i'-1,\kappa'}^{(N')})^s}{s!} \left(\frac{\partial^{r+s}}{\partial z_1{}^r \partial z_2{}^s} F(z_1, z_2) \right)_{\substack{z_1 = \eta_{i-1,\kappa}^{(N)} \\ z_2 = \xi_{i'-1,\kappa}^{(N')}}} \tag{A.2.5}$$

where

$i = 0, \ldots, l$
$i' = 0, \ldots, l'$
$\kappa = 1, \ldots, N$
$\kappa' = 1, \ldots, N'$.

Taking repeatedly into account equation (A.2.5) and the analogous ones for the partial derivatives, and suitably truncating these series at each stage, one obtains

$$F(\eta_\kappa^{(N)}, \xi_{\kappa'}^{(N')}) = \lim_{\substack{n\to\infty \\ n'\to\infty}} \sum_{r=0}^{N} \sum_{s=0}^{N'} a_{r\kappa}^{(N)} b_{s\kappa'}^{(N')} \cdot \left(\frac{\partial^{r+s}}{\partial z_1{}^r \partial z_2{}^s} F(z_1, z_2) \right)_{z_1 = \eta_{0\kappa}^{(N)},\, z_2 = \xi_{0\kappa'}^{(N')}} \tag{A.2.6}$$

where $a_{r\kappa}^{(N)}, b_{s\kappa'}^{(N')}$ are suitable polynomials respectively of the variables $\eta_{i\kappa}^{(N)} - \eta_{i-1,\kappa}^{(N)}$, $i = 1, \ldots, l$ and of the variables $\xi_{i',\kappa'}^{(N')} - \xi_{i'-1,\kappa'}^{(N')}$, $i' = 1, 2, \ldots, l'$.

By (A.2.4) and (A.2.6) one has

$$\int_{C_1} dz_1 \int_{C_2} dz_2 (\overleftarrow{G}_1(z_1^*) f, A\overleftarrow{G}_2(z_2) g)_{\bar{h}}$$

$$= \lim_{\substack{N\to\infty \\ N'\to\infty}} \left(\sum_{\kappa=1}^{N} \sum_{r=0}^{n(N)} a_{r\kappa}^{*(N)} \Delta\eta_\kappa^{(N)*} \left(\frac{d^r \overleftarrow{G}_1(z_1)}{dz_1{}^r} \right)_{z_1 = \eta_\kappa^{(N)*}} f \right. \\ \left. A \sum_{\kappa'=1}^{N'} \sum_{s=0}^{n'(N')} b_{s\kappa}^{(N')} \Delta\xi_{\kappa'}^{(N')} \left(\frac{d^s \overleftarrow{G}_2(z_2)}{dz_2{}^s} \right)_{z_2 = \xi_{\kappa'}^{(N')}} g \right)_{\bar{h}}$$

from which

$$\overleftarrow{G}_{1(-)N} = \sum_{\kappa=1}^{N} \sum_{r=0}^{n(N)} a_{r\kappa}^{*(N)} \Delta\eta_{\kappa}^{(N)*} \left(\frac{d^r \overleftarrow{G}_1(z)}{dz^r} \right)_{z=\eta_\kappa^{(N)*}}$$

$$\overleftarrow{G}_{2(+)N'} = \sum_{\kappa=1}^{N'} \sum_{s=0}^{n'(N')} b_{s\kappa'}^{(N')} \Delta\xi_{\kappa'}^{(N')} \left(\frac{d^s \overleftarrow{G}_2(z)}{dz^s} \right)_{z=\xi_{\kappa'}^{(N')}} \tag{A.2.7}$$

One meets expression (A.2.2) in the embedding problem of a one unstable particle system (see B) and also in the problem of defining decay amplitudes (see Section 6), in which case $\overleftarrow{G}_1(z)$ is independent of z.

In the former case (A.2.2) shows the following structure:

$$\overleftarrow{G}_1(z) = -\frac{1}{2\pi i} \frac{1}{z-\lambda^*} \overleftarrow{G}_{(-)0}(z), \qquad \overleftarrow{G}_2(z) = \frac{1}{2\pi i} \frac{1}{z-\lambda} \overleftarrow{G}_{(+)0}(z),$$

$$A = e^{-i\bar{E}_0 t} \tag{A.2.8}$$

with $\overleftarrow{G}_{(\pm)0}(z)$ defined by (4.5) and (4.6); $C_1 = C_2 = C$ is a circuit around $z = \lambda$.

Putting

$$F(z_1, z_2, t) = \left(\frac{1}{2\pi i}\right)^2 \frac{1}{z_1 - \lambda} \frac{1}{z_2 - \lambda} \cdot (\overleftarrow{G}_{(-)0}(z_1^*)\bar{u}, \exp[-i\bar{E}_0 t]\, \overleftarrow{G}_{(+)0}(z_2)\bar{u})_{\bar{h}_0} \tag{A.2.9}$$

one has, by equation (5.30) of B,

$$\begin{aligned} F(z_1, z_2, t) = {} & \frac{Z}{(2\pi i)^2} \frac{1}{z_1 - \lambda} \frac{1}{z_2 - \lambda} \\ & \cdot \Bigg\{ e^{-iz_2 t} \left(\bar{T}_0 \bar{u}, \left[1 + \frac{\bar{M}_{0(+)}(z_1) - \bar{M}_{0(+)}(z_2)}{z_1 - z_2} \right] \bar{u} \right)_{\bar{h}_0} \\ & - \frac{1}{2\pi i} \int_{-\infty + i\epsilon}^{+\infty + i\epsilon} dz\, e^{-izt} \frac{1}{z_1 - \lambda} \frac{1}{z - z_2} \Bigg[\Bigg([z_1^* + \bar{M}_{0+}^{+}(z_1^*)]\, \bar{T}_0 \bar{u}, \\ & \frac{1}{z + \bar{M}_0(z)} [z_2 + \bar{M}_{0(+)}(z_2)]\, \bar{u} \Bigg)_{\bar{h}_0} - (\bar{T}_0 \bar{u}, [z_2 + \bar{M}_{0(+)}(z_2)]\, \bar{u})_{\bar{h}_0} \Bigg] \Bigg\} \end{aligned} \tag{A.2.10}$$

Therefore for $0 \leqslant t < T$

$$|F(z_1, z_2, t)| \leqslant K$$

$$K = \frac{|Z|}{(2\pi)^2}\left\{ e^{\eta T} \sup_{\substack{z_1 \in C \\ z_2 \in C}} \left| \frac{1}{z_1 - \lambda} \frac{1}{z_2 - \lambda} \left(\bar{T}_0 \bar{u} \left[1 + \frac{\bar{M}_{0(+)}(z_1) - \bar{M}_{0(+)}(z_2)}{z_1 - z_2} \right] \bar{u} \right)_{\bar{h}_0} \right| + \frac{e^{\epsilon T}}{\epsilon} \right.$$

$$\cdot \left[\sup_{\substack{z_1 \in C \\ z_2 \in C}} \int_{-\infty + i\epsilon}^{+\infty + i\epsilon} ds \frac{1}{|z_1 - z|} \frac{1}{|z - z_2|} \right] \left[\sup_{z_1 \in C} \left\| \frac{\overset{*}{z_1} + \bar{M}^{+}_{0(+)}(z_1)}{z_1 - \lambda} \bar{T}_0 \bar{u} \right\|_{\bar{h}_0} \right]$$

$$\cdot \left[\sup_{z_2 \in C} \left\| \frac{z_2 + \bar{M}_{0(+)}(z_2)}{z_2 - \lambda} \bar{u} \right\|_{\bar{h}_0} \right] + e^{\epsilon T}$$

$$\cdot \left[\sup_{\substack{z_1 \in C \\ z_2 \in C}} \frac{1}{|z_1 - \lambda|} \int_{-\infty + i\epsilon}^{+\infty + i\epsilon} ds \frac{1}{|z_1 - z|} \frac{1}{|z - z_2|} \right]$$

$$\left. \cdot \left[\sup_{z_2 \in C} \left\| \frac{z_2 + \bar{M}_{0(+)}(z_2)}{z_2 - \lambda} \bar{u} \right\|_{\bar{h}_0} \right] \right\} \qquad \text{(A.2.11)}$$

where $\eta = \sup_{z \in C} \operatorname{Im} z$ and we have taken into account that (Lanz, Lugiato & Ramella, 1971) $\| [z + \bar{M}_0(z)]^{-1} \|_{\bar{h}_0} \leqslant 1/\operatorname{Im} z$ for $\operatorname{Im} z > 0$.

By the uniform boundness of $F(z_1, z_2, t)$ with respect to t for $0 \leqslant t < T$, also the convergence in (A.2.4), (A.2.5) and finally the convergence in (A.2.7) is uniform with respect to t, $0 \leqslant t < T$. Therefore for any $\epsilon > 0$ one can find $\overleftarrow{G}_{(-)}$, $\overleftarrow{G}_{(+)}$ such that

$$|(f, V(t)g)_{\mathscr{H}} - (\overleftarrow{G}_{(-)} f, e^{-i\bar{H}t} \overleftarrow{G}_{(+)} g)_{\bar{\mathscr{H}}}| < \epsilon \qquad \text{(A.2.12)}$$

for all $f, g \in \mathscr{H}$, $0 \leqslant t < T$, where

$$\overleftarrow{G}_{(-)}\{f(\mathbf{p})\} = \{\overleftarrow{G}_{(-)0} f(\mathbf{p})\}, \qquad \overleftarrow{G}_{(+)}\{g(\mathbf{p})\} = \{\overleftarrow{G}_{(+)0} g(\mathbf{p})\}$$

and $\overleftarrow{G}_{(-)0} = \overleftarrow{G}_{(-)0N}$, $\overleftarrow{G}_{(+)0} = \overleftarrow{G}_{(+)0N}$, with N large enough.

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